To appear in *Potential analysis*

Sharp Growth Estimates for Modified Poisson Integrals in a Half Space

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November 22, 2000

Abstract. For continuous boundary data, including data of polynomial growth, modified Poisson integrals are used to write solutions to the half space Dirichlet and Neumann problems in $\mathbb{R}^n$. Pointwise growth estimates for these integrals are given and the estimates are proved sharp in a strong sense. For decaying data, a new type of modified Poisson integral is introduced and used to develop asymptotic expansions for solutions of these half space problems.

1. Introduction.

For x in the half space $\Pi_+ = \{x \in \mathbb{R}^n | x_n > 0\}$ $(n \geq 2)$ let $y \in \mathbb{R}^{n-1}$ be identified with the projection of x onto the hyperplane $\partial\Pi_+$. Let $B_r(a)$ be the open ball in $\mathbb{R}^{n-1}$ with centre $a \in \mathbb{R}^{n-1}$ and radius $r > 0$. When $a = 0$ we write B_r. And let θ be the angle between x and $\hat{e}_n$, i.e., $x_n = |x| \cos\theta$, $|y| = |x| \sin\theta$ and $0 \leq \theta < \pi/2$ when $x \in \Pi_+$. We will write $x = \sum_{i=1}^{n-1} y_i \hat{e}_i + x_n \hat{e}_n$ where $\hat{e}_i$ is the i^{th} unit coordinate vector and $\hat{e}_n$ is normal to $\partial\Pi_+$. Unit vectors will be denoted with a caret, $\hat{x} = x/|x|$ for $x \neq 0$.

[1] Research partially supported by an NSERC Individual Research Grant.
[2] Research partially supported by an Ontario Graduate Scholarship.

1991 *Mathematics Subject Classification.* Primary 31B10, Secondary 35J05.
Key words and phrases. Poisson integral, half space Dirichlet problem, half space Neumann problem.

For $\lambda > 0$ ($\lambda \in \mathbb{R}$) and $y' \in \mathbb{R}^{n-1}$ define the kernel

$$K(\lambda, x, y') = \left[|y' - y|^2 + x_n^2\right]^{-\lambda}. \tag{1.1}$$

The Poisson integrals for the half space problem $\Delta u = 0$ ($x \in \Pi_+$) with Dirichlet and Neumann data $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ on $\partial\Pi_+$ are, respectively,

$$D[f](x) = \alpha_n x_n \int\limits_{\mathbb{R}^{n-1}} f(y') K\left(\tfrac{n}{2}, x, y'\right) \, dy' \quad (n \geq 2) \tag{1.2}$$

and

$$N[f](x) = \frac{\alpha_n}{n-2} \int\limits_{\mathbb{R}^{n-1}} f(y') K\left(\tfrac{n-2}{2}, x, y'\right) \, dy' \quad (n \geq 3). \tag{1.3}$$

Here $\alpha_n = 2/(n\omega_n)$ and $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$ is the volume of the unit n-ball . When $n = 2$, N has a logarithmic kernel and is not dealt with here. (See the remarks at the end of this section.)

The functions defined by (1.2) and (1.3) will be harmonic in Π_+ if

$$\int\limits_{\mathbb{R}^{n-1}} \frac{|f(y')| \, dy'}{|y'|^{2\lambda} + 1} < \infty \tag{1.4}$$

with $\lambda = n/2$ and $(n-2)/2$, respectively ([9], Theorem 6). If f is continuous then convergence of the appropriate integral in (1.4) is sufficient for (1.2) or (1.3) to be a classical solution of the respective Dirichlet or Neumann problem on Π_+ (cf. Corollary 2.1 and 2.2 below). Notice that since Π_+ is unbounded the integral of f over $\mathbb{R}^{n-1}$ need not vanish for $N[f]$ to be a solution of the Neumann problem.

When the integral in (1.4) diverges but

$$\int\limits_{\mathbb{R}^{n-1}} \frac{|f(y')| \, dy'}{|y'|^{M+2\lambda} + 1} < \infty \tag{1.5}$$

for a positive integer M we can use the modified kernel

$$K_M(\lambda, x, y') = K(\lambda, x, y') - \sum_{m=0}^{M-1} \frac{|x|^m}{|y'|^{m+2\lambda}} C_m^\lambda(\sin\theta \cos\theta') \tag{1.6}$$

(defined for $|y'| > 0$) where $0 \leq \theta' \leq \pi$ is the angle between y and y', i.e., $y \cdot y' = |y'| \, |x| \sin\theta \cos\theta'$ and $K_0 = K$. If $y = 0$ or $y' = 0$ we take $\theta' = \pi/2$. When $n = 2$, we take $\theta' = 0$ or π according as y' and x_1 are on the same or opposite side of the origin. Equivalently, $\cos\theta' = \mathrm{sgn}(x_1 \, y')$. In (1.6) the first M terms of the asymptotic

expansion of K in inverse powers of $|y'|$ are removed. The coefficients are in terms of Gegenbauer polynomials, C_m^λ, most of whose properties used herein are derived in [16].

Let $w\colon \mathbb{R}^{n-1} \to [0,1]$ be continuous so that $w(y) \equiv 0$ for $0 \le |y| \le 1$ and $w(y) \equiv 1$ for $|y| \ge 2$. Define modified Dirichlet and Neumann integrals

$$D_M[f](x) = \alpha_n x_n \int\limits_{\mathbb{R}^{n-1}} f(y') K_M\left(\tfrac{n}{2}, x, y'\right) \, dy' \quad (n \ge 2) \tag{1.7}$$

$$N_M[f](x) = \frac{\alpha_n}{n-2} \int\limits_{\mathbb{R}^{n-1}} f(y') K_M\left(\tfrac{n-2}{2}, x, y'\right) \, dy' \quad (n \ge 3). \tag{1.8}$$

Then $u(x) = D_M[wf](x) + D[(1-w)f](x)$ and $v(x) = N_M[wf](x) + N[(1-w)f](x)$ are respective solutions of the classical half space Dirichlet and Neumann problems. The purpose of the function w is merely to avoid the singularity of the modified kernel at the origin. The Dirichlet version of K_M appears in [4], [15] and [18], with inspiration from [8]. The Neumann version is discussed by Gardiner ([10]) and Armitage ([3]).

In this paper we give growth estimates for u and v under (1.5) and prove they are sharp. This is done in Theorem 2.1 by first defining

$$F_{\lambda,M}[f](x) = \int\limits_{|y'|>1} f(y') K_M(\lambda, x, y') \, dy' \tag{1.9}$$

and proving that

$$F_{\lambda,M}[f](x) = o(|x|^M \sec^{2\lambda}\theta) \quad \text{as} \quad |x| \to \infty \quad \text{with} \quad x \in \Pi_+. \tag{1.10}$$

The order relation is interpreted as $\mu(r) r^{-M} \to 0$ as $r \to \infty$ where $\mu(r)$ is the supremum of $|F_{\lambda,M}[f](x)| \cos^{2\lambda}\theta$ over $x \in \Pi_+$, $|x| = r$. A growth condition ω is said to be sharp if given any function $\psi = o(\omega)$ and any sequence $\{x_i\} \in \Pi_+$ with $|x_i| \to \infty$, we can find data f so that the solution corresponding to f is not $o(\psi)$ on this sequence (see Definition 2.1 below). The sharpness proof is complicated by the fact that the modified kernels are not of one sign. For each x_i, regions in $\mathbb{R}^{n-1}$ are determined where the kernel is of one sign. Data is then chosen so that the contribution from integrating where the sign of the kernel is not known is cancelled out and the main contribution comes from integrating over a neighbourhood of the singularity of the kernel. This proof makes up a substantial portion of the paper. Note that condition (1.5) is necessary and sufficient for $F_{\lambda,M}[f](x)$ to exist as a Lebesgue integral on Π_+. See Proposition 3.4.1 in [17].

In the third section of the paper, the modified Neumann operator N_M is represented as various integrals of the modified Dirichlet operator D_M.

In the final section a second type of modified kernel is defined, (4.1), useful when

$$\int\limits_{\mathbb{R}^{n-1}} |f(y')|\,(|y'|^{M-1}+1)\,dy' < \infty \tag{1.11}$$

for a positive integer M. This new kernel will be used in Theorem 4.2 to derive asymptotic expansions of $D[f]$ and $N[f]$ under (1.11). A growth estimate is given for the remainder term and the estimate is proved sharp as in the previous case.

The half plane ($n = 2$) Neumann Poisson integral has a logarithmic kernel. A modified kernel can be defined as in (1.6) and is in some sense the limit as $\lambda \to 0^+$. However, this case is sufficiently different to warrant separate exposition. Analogues of the results in this paper will be presented elsewhere.

2. First type of modified kernel.

The expansion (1.6) arises from the generating function for Gegenbauer polynomials ([16], 4.7.23)

$$(1-2tz+z^2)^{-\lambda} = \sum_{m=0}^{\infty} z^m C_m^\lambda(t), \quad \lambda > 0, \tag{2.1}$$

where $C_m^\lambda(t) = \frac{1}{m!}\frac{\partial^m}{\partial z^m}(1-2tz+z^2)^{-\lambda}\Big|_{z=0}$. If $-1 \le t \le 1$ the series converges absolutely for $|z| < 1$ (the left side of (2.1) is singular at $z = t \pm i\sqrt{1-t^2}$). The majorisation and derivative formulas

$$|C_m^\lambda(t)| \le C_m^\lambda(1) = \binom{2\lambda+m-1}{m} = \frac{\Gamma(2\lambda+m)}{\Gamma(2\lambda)\,\Gamma(m+1)} \tag{2.2}$$

$$\frac{d}{dt}C_m^\lambda(t) = 2\lambda\, C_{m-1}^{\lambda+1}(t) \tag{2.3}$$

are proved in [16] (4.7.3, 7.33.1, 4.7.14). Hence, the series in (2.1) converges if $|z| < 1$, uniformly for $-1 \le t \le 1$ and the same can be said for all of its derivatives with respect to z and t. From the definition above and Faà di Bruno's formula for the m^{th} derivative of a composite function ([1], p. 823) it can be seen that $C_m^\lambda(t)$ is a polynomial in t of degree m. And,

$$C_0^\lambda(t) = 1, \quad C_1^\lambda(t) = 2\lambda t. \tag{2.4}$$

A proof of the following lemma is hinted at in [4], [10] and [18] by reference to a more general result on axial polynomials in [13] (Theorem 2). However, we give a simple direct proof.

Lemma 2.1 *For $m = 0, 1, 2, 3, \ldots$ the functions $h^{(0)}_{m+1}(x) = x_n|x|^m\ C^{n/2}_m(\Theta)(n \geq 2)$ and $h^{(1)}_m(x) = |x|^m C^{(n-2)/2}_m(\Theta)$ $(n \geq 3)$ are homogeneous harmonic polynomials of degree $m+1$ and m, respectively, where $\Theta = \sin\theta\cos\theta'$.*

Proof: Using (2.1) we obtain the expansion of the fundamental solution of Laplace's equation

$$|x - x'|^{2-n} = \sum_{m=0}^{\infty} \frac{|x|^m}{|x'|^{m+n-2}} C^{(n-2)/2}_m(\hat{x}\cdot\hat{x}'), \quad n \geq 3. \tag{2.5}$$

If $x' \neq 0$ this series converges for $|x| < |x'|$ and defines a harmonic function. Each term is homogeneous in x of degree m and it is clear from (2.4) that the first two terms are harmonic. Given x, take x' such that $|x'| > |x|$. Differentiating termwise in x gives

$$\Delta|x - x'|^{2-n} = 0 = \sum_{m=2}^{\infty} |x'|^{-(m+n-2)} \Delta\left(|x|^m C^{(n-2)/2}_m(\hat{x}\cdot\hat{x}')\right). \tag{2.6}$$

Each term $\Delta\left(|x|^m C^{(n-2)/2}_m(\hat{x}\cdot\hat{x}')\right)$ is homogeneous of degree $m-2$, hence, by the linear independence of homogeneous functions, $|x|^m C^{(n-2)/2}_m(\hat{x}\cdot\hat{x}')$ is harmonic on $\mathbb{R}^n$ for each $m \geq 0$. Every harmonic function can be uniquely written as a sum of homogeneous harmonic polynomials so $|x|^m\, C^{(n-2)/2}_m(\hat{x}\cdot\hat{x}')$ is a homogeneous harmonic polynomial of degree m ([5], 1.26, 1.27).

Now set $x'_n = 0$, then $\hat{x}\cdot\hat{x}' = y\cdot y'/(|x|\,|y'|) = \sin\theta\ \hat{y}\cdot\hat{y}' = \Theta$. Hence, the Neumann half space expansion is

$$\left[|y' - y|^2 + x_n^2\right]^{-(n-2)/2} = \sum_{m=0}^{\infty} \frac{|x|^m}{|y'|^{m+n-2}} C^{(n-2)/2}_m(\Theta), \quad n \geq 3, \tag{2.7}$$

and each term in the series is a homogeneous harmonic polynomial of degree m.

For the Dirichlet expansion differentiate (2.5) with respect to x'_n, use (2.3) and (2.4), and set $x'_n = 0$. Then for $n \geq 3$

$$x_n\left[|y' - y|^2 + x_n^2\right]^{-n/2} = \sum_{m=0}^{\infty} \frac{x_n|x|^m}{|y'|^{m+n}} C^{n/2}_m(\Theta), \tag{2.8}$$

and each term in the series is a homogeneous harmonic polynomial of degree $m+1$.

When $n = 2$, use $C^1_m(\cos\phi) = \sin[(m+1)\phi]\csc\phi$. Then from (2.5) we recover the trigonometric expansion

$$\frac{x_2}{(\xi - x_1)^2 + x_2^2} = \sum_{m=1}^{\infty} \frac{r^m \sin(m\phi)}{\xi^{m+1}}, \quad r < |\xi|, \tag{2.9}$$

where we have written $r = |x|$, and $\phi = \pi/2 - \theta$ to conform with the usual polar coordinates ($x_1 = r\cos\phi$, $x_2 = r\sin\phi$). Each $r^m \sin(m\phi)$ is a homogeneous harmonic polynomial of degree m. ■

Remark 2.1 When $x_n = 0$, $h_m^{(0)}$ and $\partial h_m^{(1)}/\partial x_n$ vanish. The spherical harmonics of degree m are the restriction of the homogeneous harmonic polynomials to the unit sphere. If we write $Y_m^{(0)}(\hat{x}) = h_m^{(0)}(\hat{x})$ and $Y_m^{(1)}(\hat{x}) = h_m^{(1)}(\hat{x})$ then $h_m^{(0)}(x) = |x|^m Y_m^{(0)}(\hat{x})$ and $h_m^{(1)}(x) = |x|^m Y_m^{(1)}(\hat{x})$. The functions $|x|^{-(m+n-2)} Y_m^{(0)}(\hat{x})$ and $|x|^{-(m+n-2)} Y_m^{(1)}(\hat{x})$ are harmonic for $|x| > 0$ (interchange x and x' in (2.5) and (2.6)).

Our first theorem will be a sharp growth estimate for $F_{\lambda,M}$. First we introduce the following definition.

Definition 2.1 *Let $\omega\colon \Pi_+ \to (0,\infty)$ then ω is a sharp growth condition for $F_{\lambda,M}$ if*

(i) $F_{\lambda,M}[f](x) = o(\omega(x))$ $(x \in \Pi_+,\ |x| \to \infty)$, *for all f satisfying* (1.5)

(ii) If ψ: $\Pi_+ \to (0,\infty)$ and $\psi(x) = o(\omega(x))$ then for any sequence $\{x^{(i)}\}$ in Π_+ such that $|x^{(i)}| \to \infty$ as $i \to \infty$ there exists a continuous function f satisfying (1.5) *with $F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \not\to 0$ as $i \to \infty$.*

Note that it is essential that the limit condition on $F_{\lambda,M}[f]/\psi$ be checked on all paths to infinity. For example, $\omega_1(x) = |x|$ and $\omega_2(x) = |x|\sec\theta$ agree on all radial paths but allow very different behaviour on paths approaching $\partial\Pi_+$.

Let

$$\Phi_\pm(\Theta,\zeta) = M C_M^\lambda(\Theta) \pm (2\lambda + M - 1) C_{M-1}^\lambda(\Theta)\,\zeta \quad \text{and} \quad \Theta = \sin\theta\cos\theta'. \tag{2.10}$$

The integral representation for $M \geq 1$

$$K_M(\lambda,x,y') = K(\lambda,x,y') \int_{\zeta=0}^{|x|/|y'|} (1 - 2\Theta\zeta + \zeta^2)^{\lambda-1} \Phi_-(\Theta,\zeta)\,\zeta^{M-1}\,d\zeta \tag{2.11}$$

was derived in [15] (Theorem 5.1) by summing a Gegenbauer recurrence relation. Use of (2.11) allows us to prove

Theorem 2.1 *Let $\lambda > 0$ and f be measurable so that* (1.5) *holds for integer $M \geq 0$. Then $F_{\lambda,M}[f](x) = o\left(|x|^M \sec^{2\lambda}\theta\right)$ $(x \in \Pi^+, |x| \to \infty)$ and the order relation is sharp in the above sense.*

A weaker form of sharpness (with respect to the exponents of $|x|$ and $\sec\theta$) was obtained for $D[f]$ in [15]. Also, a growth estimate like (1.10) was obtained for $D_M[f]$ but we provide a shorter proof here. Sharpness of the order relation is proven by finding regions in $\mathbb{R}^{n-1}$ where $K_M(\lambda, x, y')$ is of one sign (for fixed x). Data f is then chosen large enough so that $F_{\lambda,M}(x)$ is positive for all x and equal to $\psi(x)$ on a subsequence of the given sequence in (ii) of Definition 2.1. The proof is quite long but has been broken down into digestible pieces as detailed below. Of crucial importance is the integral form of the modified kernel, given in (2.11).

Step I It is shown that $F_{\lambda,M}[f] = o(|x|^M \sec^{2\lambda}\theta)$ for any measurable function f satisfying the integrability condition (1.5). In (2.11), the original kernel K is estimated as was done in Theorem 2.1 of [15] ($\alpha = n/2$). The function $\Phi(\Theta, \zeta)$ has a simple zero precisely where $1 - 2\theta\zeta + \zeta^2$ vanishes, at $\Theta = \zeta = 1$. So the ratio $\Phi(\Theta, \zeta)/\sqrt{1 - 2\theta\zeta + \zeta^2}$ is bounded and the integrand in (2.11) is continuous for $\lambda \geq 1/2$ and unbounded but integrable when $0 < \lambda < 1/2$. In either case, elementary approximations lead to an upper bound for $|K_M|$ on which the Dominated Convergence Theorem can be used to prove (1.10).

Step II The order estimate $o(|x|^M \sec^{2\lambda}\theta)$ is now proven to be sharp, first for given sequences which have a subsequence $\tilde{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$ that stays bounded away from the $\hat{e}_1$ axis of $\partial\Pi_+$ by an angle θ_0 ($0 \leq \theta_0 < \pi/2$). On such a sequence the growth condition reduces to $o(|x|^M)$. A region $\Omega_1 \subset \mathbb{R}^{n-1}$ is found on which Φ_- and hence K_M are of one sign. Due to the parity of C_m^λ about zero (C_m^λ is even if m is even and odd if m is odd) it follows that $\Phi_-(\Theta, \zeta)$ will be of one sign if $|\Theta|$ is small enough. Since $\Theta = \sin\theta\cos\theta'$, this is accomplished by restricting θ' to lie near $\pi/2$. And, Ω_1 is taken as the region between two cones, both of which have an opening angle of nearly $\pi/2$ from the $\hat{e}_1$ axis. For $y' \in \Omega_1$, the combination $(-1)^{\lceil M/2\rceil}\Phi_-(\Theta, \zeta)$ is strictly positive when $\zeta > 0$. (When $x \in \mathbb{R}$, the *ceiling* of x, $\lceil x\rceil$, is x if $x \in \mathbb{Z}$ and is the next largest integer if $x \notin \mathbb{Z}$.) A lower bound on $(-1)^{\lceil M/2\rceil}K_M$ is now obtained, equation (2.26). Data is then chosen that has support in Ω_1 and is large on a sequence of unit half balls along the $\hat{e}_2$ axis. (This is an axis orthogonal to $\hat{e}_1$. Something slightly different is done when $n = 2$.) By taking the data large enough we have $\limsup F_{\lambda,M}[f]/\psi \geq 1$ on a subsequence and sharpness of the growth estimate for this special type of sequence now follows.

Step III Now considered are sequences with a subsequence $\tilde{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$ that approaches the boundary at the $\hat{e}_1$ axis. Again, a region is found where Φ_- is of one sign. On the sequence, we have $\sin\theta \to 1$ so taking θ' near 0 makes Θ nearly equal to 1. In this case then, the kernel $K(\lambda, x, y')$ will be singular for $|y'| = |x|$ and $\Theta \to 1$. Hence, in (1.6) it will dominate the Gegenbauer terms subtracted from it. A region $\Omega_2 \subset \mathbb{R}^{n-1}$ is defined to be the portion of a cone with $|y'| > 1$ and axis along $\hat{e}_1$. The opening angle θ' is taken small enough so that when $y'_1 > 0$, $|y'|$ is near $|x|$ and $|x|/A < |y'| < A|x|$ for a constant $A > 1$, we have $K_M > 0$, i.e., near the singularity of K. The modified kernel is also positive for large values of

$|y'|$ in Ω_2 but changes sign when $y_1' > 0$ and $1 < |y'| < |x|/A$. And, due to the parity of C_m^λ, the modified kernel is one sign when $y' \in \Omega_2$ with $y_1' < 0$. Data is chosen to have support within Ω_2 on a sequence of balls along the $\hat{e}_1$ axis. When y_1' is positive, $f(y')$ is positive and when y_1' is negative, fK_M is positive. Contributions to $\int_{|y'|>1} f(y')K_M(\lambda, x, y')\, dy'$ are now known to be positive except when integrating over $\Omega_>$, that portion of Ω_2 with $1 < |y'| < |x|/A$. But f is chosen so that if the reflection of y' across the $y_1' = 0$ hyperplane is denoted y^*, then if $y_1' > 0$ we have $f(y^*) = (-1)^M A_\lambda f(y')$, where $A_\lambda > 1$ is a constant. The data is said to be given a "super odd" or "super even" extension from $y_1' > 0$ to $y_1' < 0$, according as M is even or odd. This allows the contribution from integrating over $\Omega_>$, where fK_M is not of one sign, to be balanced out by the contribution from integrating over the reflection of $\Omega_>$ to $y_1' < 0$, where fK_M is positive. The contribution to $\int_{|y'|>1} f(y')K_M(\lambda, x, y')\, dy'$ from integrating near the singularity of K_M, i.e., over Ω_2, produces a lower bound for $F_{\lambda,M}[f]$ from which it follows that $F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \not\to 0$, where ψ and $x^{(i)}$ are given in the theorem. Note that all the Ω regions defined here depend on $|x|$.

Step IV The special case of sequences $\tilde{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$ considered in II and III is shown to be applicable to general sequences in Π_+. Since ∂B_+ is compact, for any sequence $r_i\hat{r}_i$ in Π_+, the sequence $\{\hat{r}_i\}$ has a limit point $\hat{r}_0 \in \partial\overline{B}_+$. This direction is then rotated to correspond to $\hat{y}_1$.

Proof: Write $s = |x|/|y'|$. Throughout the proof $d_1, d_2, \ldots, d_9$ will be positive constants (depending on λ and M).

Step I First suppose $M \geq 1$.

In [16] (4.7.27) for $M \geq 2$ we have

$$MC_M^\lambda(t) = (2\lambda + M - 1)\, t\, C_{M-1}^\lambda(t) - 2\lambda\, (1 - t^2)\, C_{M-2}^{\lambda+1}(t). \tag{2.12}$$

With reference to (2.10) and (2.2) we can write

$$\begin{aligned}
\frac{|\Phi_-(\Theta,\zeta)|}{\sqrt{1 - 2\Theta\zeta + \zeta^2}} &= \frac{\left|(2\lambda + M - 1)(\Theta - \zeta)C_{M-1}^\lambda(\Theta) - 2\lambda(1 - \Theta^2)C_{M-2}^{\lambda+1}(\Theta)\right|}{\sqrt{(\Theta - \zeta)^2 + (1 - \Theta^2)}} &(2.13)\\
&\leq (2\lambda + M - 1)\binom{2\lambda + M - 2}{M - 1} + 2\lambda\binom{2\lambda + M - 1}{M - 2}\\
&= 2\lambda\binom{2\lambda + M}{M - 1} &(2.14)
\end{aligned}$$

for $M \geq 2$. If we define $C_{-m}^\lambda = 0$ for $m = 1, 2, 3, \ldots$ and use the fact that $C_0^\lambda(\Theta) = 1$ and $C_1^\lambda(\Theta) = 2\lambda\Theta$ then (2.12) and (2.14) still hold when $M = 1$. Hence, (2.11) and (2.14) give

$$|K_M(\lambda, x, y')| \leq d_1\, K(\lambda, x, y')\, s^{M-1} \int_{\zeta=0}^{s} (1 - 2\Theta\zeta + \zeta^2)^{\lambda - \frac{1}{2}}\, d\zeta. \tag{2.15}$$

For $M \geq 1$ and $\lambda \geq \frac{1}{2}$ the integrand in (2.15) is continuous and $|\Theta| \leq 1$ so $(1-2\Theta\zeta+\zeta^2) \leq (1+s)^2$. Therefore,

$$|K_M(\lambda, x, y')| \leq d_1\, K(\lambda, x, y')\, s^M\, (1+s)^{2\lambda-1}. \tag{2.16}$$

The estimate

$$|K(\lambda, x, y')| \leq 2^{2\lambda} \sec^{2\lambda}\theta\, (|x|+|y'|)^{-2\lambda} \tag{2.17}$$

is in [15] (Corollary 2.1). Hence,

$$\begin{aligned} |K_M(\lambda, x, y')| &\leq d_2\, s^M \sec^{2\lambda}\theta\, |y'|^{-2\lambda}(1+s)^{-1} && (2.18)\\ &\leq d_2\, s^M \sec^{2\lambda}\theta\, |y'|^{-2\lambda}. && (2.19) \end{aligned}$$

Multiply (2.18) by $|f(y')|$ and integrate $y' \in \mathbb{R}^{n-1}$, $|y'| > 1$. Letting $|x| \to \infty$, the Dominated Convergence Theorem gives (1.10).

When $0 < \lambda < \frac{1}{2}$ the integrand in (2.15) can be singular. In this case

$$\begin{aligned} \int_{\zeta=0}^{s} (1-2\Theta\zeta+\zeta^2)^{\lambda-\frac{1}{2}} d\zeta &\leq \int_{\zeta=0}^{s} |1-\zeta|^{2\lambda-1}\, d\zeta \\ &= \frac{1}{2\lambda} \begin{cases} 1-(1-s)^{2\lambda}, & 0 \leq s \leq 1 \\ 1+(s-1)^{2\lambda}, & s \geq 1 \end{cases} \\ &\leq \left(\frac{1}{\lambda}\right) \min(s, s^{2\lambda}). && (2.20) \end{aligned}$$

And,

$$|K_M(\lambda, x, y')| \leq d_3\, s^M \sec^{2\lambda}\theta\, (|x|+|y'|)^{-2\lambda} \tag{2.21}$$

so (1.10) holds for $0 < \lambda < \frac{1}{2}$ as well.

Finally, integrating (2.17) shows (1.10) also holds when $M = 0$.

Step II We now prove sharpness. Given any sequence $\{x^{(i)}\}$ in Π_+ with $|x^{(i)}| \to \infty$ and any function $\psi(x) = o\left(|x|^M \sec^{2\lambda}\theta\right)$ we find a continuous function f satisfying (1.5) for which $F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \not\to 0$ as $i \to \infty$.

Note that (2.11) may be written

$$K_M(\lambda, x, y') = K(\lambda, x, y')\, s^M \int_{\zeta=0}^{1} (1-2\Theta s\zeta+s^2\zeta^2)^{\lambda-1}\, \Phi_-(\Theta, s\zeta)\, \zeta^{M-1}\, d\zeta. \tag{2.22}$$

Suppose first that $\{x^{(i)}\}$ has a subsequence $\tilde{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$, $i \geq 1$, where $b_i > 0$ and $0 \leq a_i \leq b_i \tan\theta_0$ for some $0 \leq \theta_0 < \pi/2$. Then $0 \leq \sin\theta = a_i/\sqrt{a_i^2+b_i^2} \leq$

$\sin\theta_0 < 1$. Since $\psi(x) = o(|x|^M \sec^{2\lambda}\theta)$ and $1 \le \sec\theta = \sqrt{a_i^2 + b_i^2}/b_i \le \sec\theta_0 < \infty$ we also have $\psi(x) = o(|x|^M)$. We may assume that $\tilde{x}^{(i)}$ have been chosen so that $\psi(\tilde{x}^{(i)}) \le |\tilde{x}^{(i)}|^M/i^2$, $i \ge 1$.

Now find a region $\Omega_1 \subset \mathbb{R}^{n-1}$ in which $\Phi_-(\Theta, s\zeta)$ is of one sign. Consider $n \ge 3$ and $M \ge 1$. Let β_1 be the smallest positive root of $\{C_M^\lambda, C_{M-1}^\lambda\}$. And, $C_m^\lambda(\Theta)$ is a polynomial in Θ of degree m with m simple zeroes in $(-1, 1)$. If $M = 1$, take $\beta_1 = 1$. So $0 < \beta_1 \le 1$. Now, C_m^λ is even or odd about the origin according as m is even or odd ([16], 4.7.4) and $(-1)^m C_{2m}^\lambda(0) > 0$ ([6], 10.9.19). Hence, one of $\{C_M^\lambda, C_{M-1}^\lambda\}$ changes sign at the origin and the other is of one sign on $(-\beta_1, \beta_1)$. Therefore, for any $0 \le \theta \le \pi/2$, $C_M^\lambda(\sin\theta\cos\theta')$ and $C_{M-1}^\lambda(\sin\theta\cos\theta')$ are each of one sign for $\arccos(\beta_1) \le \theta' \le \pi/2$. The same can be said when $\pi/2 \le \theta' \le \pi - \arccos(\beta_1)$. Write $M = 2\mu + \varepsilon_0$ where ε_0 is 0 or 1. From (2.3) we see that if $0 < t < \beta_1$ then $\mathrm{sgn}(C_{2\mu+1}^\lambda(t)) = \mathrm{sgn}(C_{2\mu}^{\lambda+1}(t)) = (-1)^\mu$ and if $-\beta_1 < t < 0$ then $\mathrm{sgn}(C_{2\mu+1}^\lambda(t)) = -\mathrm{sgn}(C_{2\mu}^{\lambda+1}(t)) = (-1)^{\mu+1}$. Let

$$\begin{aligned}\Omega_1(\hat{y}) \;=\; & \Big\{ y' \in \mathbb{R}^{n-1} \Big| \arccos(\beta_1/2) \le \theta' \le \arccos(\beta_1/3) \quad \text{if } M \text{ is even} \\ & \quad \text{and} \quad \arccos(\beta_1/3) \le \theta' \le \pi - \arccos(\beta_1/2) \quad \text{if } M \text{ is odd} \Big\}. \end{aligned} \tag{2.23}$$

Then, since C_M^λ and C_{M-1}^λ have no common roots, there exists a positive constant d_4 such that

$$(-1)^{\mu+\varepsilon_0}\Phi_-(\Theta, s\zeta) \ge d_4, \tag{2.24}$$

whenever $0 \le \theta \le \pi/2$, $y' \in \Omega_1(\hat{y})$, $0 \le \zeta \le 1$, $s \ge 0$. In (2.23), θ' is restricted to lie in a smaller region than $\arccos\beta_1 \le \theta' \le \pi/2$ so that $(-1)^{\mu+\varepsilon_0}\Phi_-$ will be strictly positive for $y' \in \Omega_1$.

From (2.22) we will need the estimate,

$$\begin{aligned}(1 - 2\Theta s\zeta + s^2\zeta^2)^{\lambda-1} \;&\ge\; \begin{cases} (1+s)^{2(\lambda-1)}, & 0 \le \lambda \le 1 \\ ((s\zeta - \sin\theta_0)^2 + \cos^2\theta_0)^{\lambda-1}, & \lambda \ge 1 \end{cases} \\ &\ge\; (1+s)^{-2}\,\cos^{2|\lambda-1|}\theta_0. \end{aligned} \tag{2.25}$$

These give

$$(-1)^{\mu+\varepsilon_0} K_M(\lambda, x, y') \ge \frac{d_5\,K(\lambda, x, y')\,s^M}{(1+s)^2}, \tag{2.26}$$

whenever $y' \in \Omega_1$.

If $M = 0$ then (2.25) and (2.26) hold and we can take $\Omega_1 = \mathbb{R}^{n-1}$.

Let

$$f(y') = \begin{cases} (-1)^{\mu+\varepsilon_0} f_i \Big[1 - |y' - c_i\hat{e}_2|\Big]|y_1'|; & y' \in B_1(c_i\hat{e}_2), (-1)^M y_1' \ge 0 \\ 0, & \text{otherwise,} \end{cases} \tag{2.27}$$

where $c_i := |\tilde{x}^{(i)}| = \sqrt{a_i^2 + b_i^2}$ and the constants f_i are defined in (2.29) below. Then $f : \mathbb{R}^{n-1} \to \mathbb{R}$, has support in a sequence of half balls along the $\hat{e}_2$ axis and is continuous. The factor $[1 - |y' - c_i\hat{e}_2|]|y_1'|$ makes f vanish on the perimeter of the i^{th} half ball. Without loss of generality we may assume $c_i \to \infty$ monotonically so that the $B_1(c_i\hat{e}_2)$ are disjoint, $\text{supp}(f) \subset \Omega_1$ and $c_i \geq 2$ (otherwise, take an appropriate subsequence of $\{\tilde{x}^{(i)}\}$).

Now, for any $j \geq 1$,

$$F_{\lambda,M}[f](\tilde{x}^{(j)}) \geq \int\limits_{B_1(c_j\hat{e}_2)} f(y')K_M(\lambda, \tilde{x}^{(j)}, y')\, dy'.$$

When $y' \in B_1(c_j\hat{e}_2)$ we have $s = |\tilde{x}^{(j)}|\,/|y'| \leq c_j/(c_j-1) \leq 2$ and $s \geq c_j/(c_j+1) \geq 2/3$. And,

$$\begin{aligned} K(\lambda, \tilde{x}^{(j)}, y') &\geq \left[(|y'| + a_j)^2 + b_j^2\right]^{-\lambda} \\ &\geq \left[(c_j + 1 + a_j)^2 + b_j^2\right]^{-\lambda} \\ &\geq (7c_j^2)^{-\lambda}. \end{aligned}$$

Thus, using (2.26),

$$\begin{aligned} F_{\lambda,M}[f](\tilde{x}^{(j)}) &\geq \frac{d_6\, f_j}{|\tilde{x}^{(j)}|^{2\lambda}} \int\limits_{B_1} (1 - |y'|)\, |y_1'|\, dy' \\ &= d_7^{-1}\, f_j\, |\tilde{x}^{(j)}|^{-2\lambda}. \end{aligned} \tag{2.28}$$

Let

$$f_i = d_7\, \psi(\tilde{x}^{(i)})\, |\tilde{x}^{(i)}|^{2\lambda}. \tag{2.29}$$

Then $f_i/c_i^{M+2\lambda} \leq (d_7 i^2)^{-1}$ and $\sum_{i=1}^{\infty} f_i c_i^{-(M+2\lambda)} < \infty$ so (1.5) holds. And, on $\{\tilde{x}^{(i)}\}$ we have $F_{\lambda,M}[f](\tilde{x}^{(j)}) \geq \psi(\tilde{x}^{(j)})$ for each $j \geq 1$ so $\limsup_{i\to\infty} F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \geq 1$ and $F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \not\to 0$ as $i \to \infty$.

When $n = 2$, write $x_1 = r\cos\phi$, $x_2 = r\sin\phi$. Then in place of (2.11) we have $F_{\lambda,M}[f](x) = \int_{-\infty}^{\infty} f(\xi)\; K_M(\lambda, x, \xi)\, d\xi$ where

$$K_M(\lambda, x, \xi) = K(\lambda, x, \xi)\left(\frac{r}{\xi}\right)^M \int\limits_{\zeta=0}^{1} \left(1 - 2\frac{r\zeta}{\xi}\cos\phi + \frac{r^2\zeta^2}{\xi^2}\right)^{\lambda-1} \Phi_-\left(\cos\phi, \frac{r\zeta}{\xi}\right)\zeta^{M-1}\, d\zeta.$$

If $0 \leq \theta \leq \theta_0 < \pi/2$ then $0 < \phi_0 \leq \phi \leq \pi - \phi_0 < \pi$ where $\phi_0 = \pi/2 - \theta_0$.

Let t_i, $1 \leq i \leq q$, be the roots of $C_M^\lambda \circ \cos$ and $C_{M-1}^\lambda \circ \cos$ in $[\phi_0,\, \pi-\phi_0]$, ordered by size. We then have the partition $\phi_0 = t_0 \leq t_1 < t_2 < \cdots < t_{q-1} < t_q \leq t_{q+1} = \pi - \phi_0$. In each interval $[t_i, t_{i+1}]$, $0 \leq i \leq q$, $C_M^\lambda \circ \cos$ and $C_{M-1}^\lambda \circ \cos$ are each of one sign. If ϕ_0 is a root, we omit the singleton $\{t_1\}$, similarly with $\pi - \phi_0$.

For any sequence $\phi_i \in [\phi_0,\, \pi-\phi_0]$, $i \geq 1$, there is a subsequence $\{\tilde{\phi}_i\}$ in one of the above intervals $[t_j, t_{j+1}]$. If $C_M^\lambda(\cos\tilde{\phi}_i)$ and $C_{M-1}^\lambda(\cos\tilde{\phi}_i)$ are of the same sign, take $\Omega_1 = \{\xi \in \mathbb{R} | \xi < 0\}$ and $\Omega_1 = \{\xi \in \mathbb{R} | \xi > 0\}$ if they are of opposite sign. Then $(-1)^{\mu_0}\Phi_-(\cos\tilde{\phi}_i, r\zeta/\xi) \geq 0$ for $i \geq 1$, $\xi \in \Omega_1$, where $(-1)^{\mu_0} = \operatorname{sgn}(C_M^\lambda(\cos\tilde{\phi}_i))$ ($\mu_0 = 0$ or 1). Since C_M^λ and C_{M-1}^λ have no common zeroes there is a subsequence $\{\check{\phi}_i\}$ of $\{\tilde{\phi}_i\}$ such that either $C_M^\lambda(\cos\check{\phi}_i)$ or $C_{M-1}^\lambda(\cos\check{\phi}_i)$ is bounded away from zero for all $i \geq 1$. Hence, there is a positive constant $\check{d}_5$ such that $(-1)^{\mu_0}\Phi_-(\cos\check{\phi}_i, \check{r}_i\zeta/\xi) \geq \check{d}_5(\check{r}_i\zeta/|\xi|)^{\mu_1}$ for $i \geq 1$. Here $\check{x}^{(i)} = \check{r}_i \cos\check{\phi}_i\, \hat{e}_1 + \check{r}_i \sin\check{\phi}_i\, \hat{e}_2$ is a sub-subsequence of the given sequence $\{x^{(i)}\}$ and μ_1 is 0 or 1. We now proceed in a similar manner to the case $n \geq 3$ given above.

Step III In the previous argument $0 \leq \theta_0 < \pi/2$ was arbitrary so now suppose that given the sequence $\{x^{(i)}\}$ there is a subsequence $\tilde{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$ such that $\sin\theta_0 \leq \sin\theta = a_i/\sqrt{a_i^2+b_i^2} < 1$. Since $0 < b_i \leq a_i \cot\theta_0$ we may assume $0 < b_i \leq a_i/2$ and that $a_i \to \infty$ monotonically.

Find a region $\Omega_2 \subset \mathbb{R}^{n-1}$ on which K_M is of one sign. Let $M \geq 1$ and let $x = x_1\hat{e}_1 + x_n\hat{e}_n \in \{\tilde{x}^{(i)}\}$. We will take $1 < A < 2$ close enough to 1 so that if

$$\frac{1}{A} \leq s \leq A, \quad \frac{1}{A} \leq \Theta \leq 1 \tag{2.30}$$

then $K_M(\lambda, x, y')$, $C_M^\lambda(\Theta)$ and $C_{M-1}^\lambda(\Theta)$ are positive. From (1.1) and (1.6), we have $K_M(\lambda, x, y') \geq |y'|^{-2\lambda}\left[(1-2A^{-2}+A^2)^{-\lambda} - \gamma_{\lambda,M}^{-\lambda}\right]$, where $\gamma_{\lambda,M} = \left(\sum_{m=0}^{M-1} 2^m C_m^\lambda(1)\right)^{-1/\lambda}$ and $0 < \gamma_{\lambda,M} < \infty$. Note that $A > 1$ implies $1 - 2A^{-2} + A^2 > 0$. Now, $K_M > 0$ if $A^4 + (1-\gamma_{\lambda,M})A^2 - 2 < 0$. Let $r_0 > 1$ be the largest root of this quartic. Let β_2 be the largest zero of $C_M^{\min(1,\lambda)}$. Then $\cos(\pi/(M+1)) \leq \beta_2 \leq \cos(\pi/(2M))$ ([16], 6.21.7). Hence, if $1 < A < \min(2, r_0, \sec(\pi/(2M)))$ and s and Θ are as in (2.30) then $K_M(\lambda, x, y') > 0$, $C_M^\lambda(\Theta) > 0$ and $C_{M-1}^\lambda(\Theta) > 0$ ([16], 6.21.3).

To satisfy $\Theta \geq 1/A$ in (2.30), we will restrict x and y' so that $\sin\theta \geq 1/\sqrt{A}$ and $\cos\theta' \geq 1/\sqrt{A}$. First, take $\theta_0 = \arcsin\left(\sqrt{A/(2A-1)}\right)$ then $\sin\theta \geq \sin\theta_0 = \sqrt{A/(2A-1)} \geq 1/\sqrt{A}$. And, since $y = x_1\hat{e}_1$, we have $\cos\theta' = (y \cdot y')/(|y|\,|y'|) = \hat{e}_1 \cdot y'\,|y'|^{-1}$ for $y' \neq 0$. Let

$$\Omega_2(\hat{y}) = \left\{y' \in \mathbb{R}^{n-1} \big| |y'| > 1,\ 1/\sqrt{A} < \cos\theta' \leq 1\right\}, \tag{2.31}$$

a portion of a cone with axis along $\hat{e}_1$. If $y' \in \Omega_2$ then $\cos\theta' \geq 1/\sqrt{A}$. If $n = 2$, take $\Omega_2 = \{\xi \in \mathbb{R} | \xi > 1\}$.

Define $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$f(y') = \begin{cases} f_i\left(1 - \frac{1}{b_i}|y' - a_i\hat{e}_1|\right),\ y' \in B_{b_i}(a_i\hat{e}_1) \quad \text{for some} \quad i \geq 1 \\ (-1)^M A_\lambda f_i\left(1 - \frac{1}{b_i}|y' + a_i\hat{e}_1|\right),\ y' \in B_{b_i}(-a_i\hat{e}_1) \quad \text{for some} \quad i \geq 1 \\ 0, \quad \text{otherwise}, \end{cases} \tag{2.32}$$

where $A_\lambda \geq 1$ is given in (2.34) and f_i in (2.40). By taking an appropriate subsequence of $\{\tilde{x}^{(i)}\}$ we may assume the balls $B_{b_i}(a_i\hat{e}_1)$ are disjoint ($a_{i+1} \geq 3a_i$ suffices). The condition $\sin\theta_0 = \sqrt{A/(2A-1)}$ ensures that each $B_{b_i}(a_i\hat{e}_1) \subset \Omega_2$. Then f is continuous, has support on a sequence of balls along the $\hat{e}_1$ axis and is non-negative for $y_1' \geq 0$.

With $y' \in \Omega_2$ such that $y_1' > 0$ and x as above (preceding (2.30)), $A^{-1} \leq \Theta = \sin\theta\cos\theta' \leq 1$. So $C_M^\lambda(\Theta)$, $C_{M-1}^\lambda(\Theta) > 0$. As a function of s, with fixed Θ as in (2.30), the integral in (2.11) is zero when $s = 0$, is an increasing function of s for $0 < s < MC_M^\lambda(\Theta)/[(2\lambda{+}M{-}1)C_{M-1}^\lambda(\Theta)]$ (where it has a maximum) and decreases for larger values of s. And, we know from the analysis following (2.30) that this integral is positive at $s = A$. Hence, $K_M(\lambda, x, y') > 0$ for $0 < s \leq A$ (with $y' \in \Omega_2$, $y_1' > 0$).

If $y' \in \Omega_2$ and $y_1' < 0$ then $\Theta < 0$. Since $C_m^\lambda(-t) = (-1)^m C_m^\lambda(t)$ we have $\Phi_-(\Theta, \zeta) = (-1)^M \Phi_+(|\Theta|, \zeta)$ and $\operatorname{sgn}(K_M(\lambda, x, y')) = (-1)^M$. From (2.32), $f(y')K_M(\lambda, x, y') \geq 0$.

Define

$$\Omega_{\gtrless} = \{y' \in \Omega_2 | y_1' \gtrless 0, \quad s > A\}. \tag{2.33}$$

If $x \in \{\tilde{x}^{(i)}\}$ then $f(y')K_M(\lambda, x, y') \geq 0$ for $y' \in \Omega_2$ except possibly for $y' \in \Omega_>$. By taking A_λ large enough we can ensure $\displaystyle\int_{\Omega_> \cup \Omega_<} f(y')K_M(\lambda, x, y')\, dy' \geq 0$. Indeed, let y^* be the reflection of y' in the hyperplane $y_1' = 0$ and θ^* the angle between y^* and y. Then $y^* \in \Omega_<$ if and only if $y' \in \Omega_>$.

If $\lambda \geq 1$ and $y^* \in \Omega_<$ then, as in (2.10), $\Theta^* := \sin\theta\cos\theta^* = -\Theta$. Then, using (2.11) and (2.32),

$$\begin{aligned} f(y^*)K_M(\lambda, x, y^*) &= A_\lambda f(y')\left[|y'|^2 + 2\Theta|y'|\,|x| + |x|^2\right]^{-\lambda} \int_{\zeta=0}^{s} \frac{\Phi_+(\Theta, \zeta)\, \zeta^{M-1}\, d\zeta}{(1 + 2\Theta\zeta + \zeta^2)^{1-\lambda}} \\ &\geq A_\lambda f(y')|x|^{-2\lambda}(1 + A^{-1})^{-2\lambda} \int_{\zeta=0}^{s} (1+\zeta^2)^{\lambda-1}\Phi_+(\Theta, \zeta)\, \zeta^{M-1}\, d\zeta. \end{aligned}$$

And,

$$f(y')K_M(\lambda,x,y') = f(y')\Big[|y'|^2 - 2\Theta|y'|\,|x| + |x|^2\Big]^{-\lambda}\int\limits_{\zeta=0}^{s}\frac{\Phi_-(\Theta,\zeta)\,\zeta^{M-1}\,d\zeta}{(1-2\Theta\zeta+\zeta^2)^{1-\lambda}}$$

$$\geq -f(y')|x|^{-2\lambda}(1-A^{-1})^{-2\lambda}\int\limits_{\zeta=0}^{s}(1+\zeta^2)^{\lambda-1}\,\Phi_+(\Theta,\zeta)\,\zeta^{M-1}\,d\zeta.$$

Therefore,

$$\int\limits_{\Omega_<\cup\Omega_>} f(y')K_M(\lambda,x,y')\,dy' \geq 0 \quad \text{if} \quad A_\lambda \geq (A+1)^{2\lambda}(A-1)^{-2\lambda}.$$

If $0<\lambda<1$ and $y^*\in\Omega_<$ then

$$f(y^*)K_M(\lambda,x,y^*) \geq A_\lambda\, f(y')\,(|x|+|y'|)^{-2\lambda}\,(1+s)^{2\lambda-2}\int\limits_{\zeta=0}^{s}\Phi_+(\Theta,\zeta)\,\zeta^{M-1}\,d\zeta$$

$$\geq A_\lambda\, f(y')\,|x|^{-2\lambda}\left(1+A^{-1}\right)^{-2\lambda} s^{M+2\lambda-1}\,\frac{(2\lambda+M-1)\,C^\lambda_{M-1}(\Theta)}{4(M+1)}.$$

If $0<\lambda<1/2$ then, using (2.14) and (2.20),

$$f(y')K_M(\lambda,x,y') \geq -f(y')(|x|-|y'|)^{-2\lambda}\,2\lambda\binom{2\lambda+M}{M-1}\frac{s^{M+2\lambda-1}}{2\lambda}.$$

And, if $1/2\leq\lambda<1$,

$$f(y')K_M(\lambda,x,y') \geq -f(y')(|x|-|y'|)^{-2\lambda}\,2\lambda\binom{2\lambda+M}{M-1}\frac{s^M}{M}\left(1+s^2\right)^{\lambda-\frac{1}{2}}.$$

Hence, for $0<\lambda<1$,

$$f(y')K_M(\lambda,x,y') \geq -2\sqrt{2}\,f(y')|x|^{-2\lambda}\left(1-A^{-1}\right)^{-2\lambda} s^{M+2\lambda-1}\binom{2\lambda+M}{M-1}.$$

And, $\displaystyle\int\limits_{\Omega_<\cup\Omega_>} f(y')K_M(\lambda,x,y')\,dy' \geq 0$ if

$$A_\lambda \geq \frac{8\sqrt{2}\,(M+1)}{2\lambda+M-1}\binom{2\lambda+M}{M-1}\left(\frac{A+1}{A-1}\right)^{2\lambda}\left[\min_{A^{-1}\leq t\leq 1} C^\lambda_{M-1}(t)\right]^{-1}.$$

Hence, for $\lambda > 0$, $x \in \{\tilde{x}^{(i)}\}$, if we take

$$A_\lambda \geq \left(\frac{A+1}{A-1}\right)^{2\lambda} \max\left(1, \frac{8\sqrt{2}\,(M+1)}{2\lambda+M-1}\binom{2\lambda+M}{M-1}\left[\min_{A^{-1}\leq t\leq 1} C^{\lambda}_{M-1}(t)\right]^{-1}\right) \tag{2.34}$$

then $F_{\lambda,M}[f](x) \geq \int\limits_{\Omega_3} f(y')K_M(\lambda,x,y')\,dy'$, where

$$\Omega_3 = \left\{y' \in \Omega_2 \middle| y'_1 > 0,\ A^{-1} < s < A\right\}. \tag{2.35}$$

Note that if $x = a_i\hat{e}_1 + b_i\hat{e}_n$ then $B_{b_i}(a_i\hat{e}_1) \subset \Omega_3$ if $a_i - |x|/A \geq b_i$ and $A|x| - a_i \geq b_i$. Since $a_i = |x|\sin\theta$, $b_i = |x|\cos\theta$ and $\theta_0 \leq \theta < \pi/2$, these conditions are satisfied if $\frac{1}{2}[\pi - \arcsin(1-A^{-2})] \leq \theta_0 < \pi/2$, i.e., by taking θ_0 close enough to $\pi/2$.

From (2.11),

$$K_M(\lambda,x,y') = K(\lambda,x,y')\,s^M \int\limits_{\zeta=0}^{1} (1-2\Theta s\zeta + s^2\zeta^2)^{\lambda-1}\,\Phi_-(\Theta,s\zeta)\,\zeta^{M-1}\,d\zeta, \tag{2.36}$$

which is strictly positive on $\overline{\Omega}_3$ ((2.30) and following). And, $K(\lambda,x,y')$ is positive but singular at $s = \Theta = 1$. Using (2.2), the integral (2.36) above reduces to

$$\frac{\Gamma(2\lambda+M)}{\Gamma(2\lambda)\,\Gamma(M)} \int\limits_{\zeta=0}^{1} (1-\zeta)^{2\lambda-1}\zeta^{M-1}\,d\zeta > 0$$

at $s = \Theta = 1$. The integral in (2.36) is a strictly positive continuous function of s and Θ when the conditions in (2.30) are satisfied. Hence, it must be bounded below by a positive constant, say d_8, i.e.,

$$K_M(\lambda,x,y') \geq d_8\,K(\lambda,x,y')\,s^M \quad \text{for} \quad y' \in \Omega_3,\ \sin\theta \geq \sin\theta_0. \tag{2.37}$$

If $M = 0$ we can dispense with the sets $\Omega_2, \Omega_3, \Omega_<$ and $\Omega_>$. In (2.32), $A_\lambda = 1$ and f is extended as an even function. Then (2.37) holds for $x \in \Pi_+$ with $d_8 = 1$.

For $M \geq 0$ each element of the sequence $\tilde{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$ satisfies $\sin\theta \geq \sin\theta_0$ so, using (2.32) and (2.37)

$$\begin{aligned}
F_{\lambda,M}[f](\tilde{x}^{(j)}) &\geq d_8|\tilde{x}^{(j)}|^M \int\limits_{\Omega_3} f(y')K(\lambda,\tilde{x}^{(j)},y')\,|y'|^{-M}\,dy' \\
&\geq d_8(a_j^2+b_j^2)^{M/2} f_j \int\limits_{B_{b_j}(a_j\hat{e}_1)} \frac{(1-|y'-a_j\hat{e}_1|\,b_j^{-1})}{(|y'-a_j\hat{e}_1|^2+b_j^2)^\lambda}\,|y'|^{-M}\,dy' \\
&\geq \frac{d_8(a_j^2+b_j^2)^{M/2}\,f_j\,b_j^{n-1}}{(a_j+b_j)^M\,b_j^{2\lambda}} \int\limits_{B_1} (1-|y'|)\,(|y'|^2+1)^{-\lambda}\,dy' \\
&\geq d_9\,f_j\,b_j^{n-1-2\lambda}
\end{aligned} \tag{2.38}$$

where $d_9 = d_8\, 2^{-M/2}(n-1)\omega_{n-1} \int\limits_{\rho=0}^{1} (1-\rho)(\rho^2+1)^{-\lambda}\, \rho^{n-2}\, d\rho$.

Note that (1.5) holds if and only if

$$\sum_{i=1}^{\infty} \frac{f_i b_i^{n-1}}{a_i^{M+2\lambda}} < \infty. \tag{2.39}$$

Now suppose $\psi\colon \mathbb{R}^{n-1} \to (0,\infty)$ such that $\psi(x) = o(|x|^M \sec^{2\lambda}\theta)$. On the sequence $\tilde{x}^{(j)} = a_j\hat{e}_1 + b_j\hat{e}_n$, $|x|^M \sec^{2\lambda}\theta = |x|^{M+2\lambda} x_n^{-2\lambda} = (a_j^2 + b_j^2)^{M/2+\lambda}\, b_j^{-2\lambda}$ and $\psi(\tilde{x}^{(j)}) = o(a_j^{M+2\lambda}\, b_j^{-2\lambda})$ (since $0 < b_i \le a_i/2$). We may assume that $\{\tilde{x}^{(i)}\}$ has been chosen so that $\psi(\tilde{x}^{(i)}) \le a_i^{M+2\lambda}\, b_i^{-2\lambda}\, i^{-2}$ for $i \ge 1$. Let

$$f_i = d_9^{-1}\, \psi(a_i\hat{e}_1 + b_i\hat{e}_n)\, b_i^{2\lambda-n+1}, \quad i \ge 1. \tag{2.40}$$

Then (2.39) is satisfied and $F_{\lambda,M}[f](\tilde{x}^{(j)}) \ge \psi(\tilde{x}^{(j)})$ so $F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \not\to 0$ as $i \to \infty$ and $F_{\lambda,M}[f](x) \ne o(|x|^M \sec^{2\lambda}\theta)$. Hence, the order relation $F_{\lambda,M}[f](x) = o(|x|^M \sec^{2\lambda}\theta)$ is sharp for $x \in \Pi_+$ of form $x = x_1\hat{e}_1 + x_n\hat{e}_n$.

Step IV For $n = 2$ this completes the proof. For $n \ge 3$ we now remove this restriction on x. For any sequence $\{x^{(i)}\}$ in Π_+, we can write $x^{(i)} = |x^{(i)}|\,\hat{x}^{(i)}$ where $\hat{x}^{(i)} \in \partial B_1^+ = \{x \in \mathbb{R}^n \big| |x| = 1,\, x_n > 0\}$. Then $\{\hat{x}^{(i)}\}$ must have a limit point, say $\hat{s}_0$, in the compact set $\overline{\partial B_1^+}$. Let θ_0 be the angle between $\hat{s}_0$ and $\hat{e}_n$.

If $0 < \theta_0 < \pi/2$ then let $\hat{s}_1$ be in the direction of the projection of $\hat{s}_0$ onto $\partial\Pi_+$ and let $\hat{s}_2 \in \partial\Pi_+$ be any unit vector orthogonal to $\hat{s}_1$. For any $\delta > 0$ there is a subsequence $\tilde{x}_\delta^{(i)} = \tilde{x}^{(i)} + \delta_i\hat{t}_i$ where $\tilde{x}^{(i)} = a_i\hat{s}_1 + b_i\hat{e}_n$, $c_i = |\tilde{x}^{(i)}| \to \infty$ monotonically, $b_i > 0$, $\{\hat{s}_1, \hat{e}_n, \hat{t}_i\}$ is orthonormal, each $\hat{t}_i \in \partial\Pi_+$ and $0 \le \delta_i \le \delta$. We can now try to repeat the first part of the sharpness proof, beginning with (2.22). Then $\hat{s}_1$ and $\hat{s}_2$ play the roles $\hat{e}_1$ and $\hat{e}_2$ did before, except that we now have perturbations by δ_i.

Let $x \in \{\tilde{x}_\delta^{(i)}\}$. Let η_i be the angle between $a_i\hat{s}_1 + \delta_i\hat{t}_i$ and $\hat{s}_1$. Without loss of generality $a_i \ge 1$. We have $0 \le \eta_i = \arctan(\delta_i/a_i) \le \delta$. Hence, we can replace (2.23) with the narrower cone

$$\begin{aligned} \Omega_1'(\hat{y}) \;=\; \Big\{ y' \in \mathbb{R}^{n-1} \Big| & \arccos(\beta_1/2) + \delta \le \theta_1' \le \arccos(\beta_1/3) - \delta \quad \text{if } M \text{ is even} \\ & \text{and } \arccos(\beta_1/3) + \delta \le \theta_1' \le \pi - \arccos(\beta_1/2) - \delta \ \text{ if } M \text{ is odd} \Big\} \end{aligned} \tag{2.41}$$

where θ_1' is the angle between y' and $a_i\hat{s}_1$. (Take $2\delta < \arccos(\beta_1/3) - \arccos(\beta_1/2)$.) For any $x \in \{\tilde{x}_\delta^{(i)}\}$, if $y' \in \Omega_1'$ then $|\cos\theta'| \le \beta_1/2$ and (2.24) holds.

If $\hat{s}_0 = \hat{e}_n$ ($\theta_0 = 0$) then take $a_i \equiv 0$. Let $\hat{s}_1 = \hat{e}_1$ and $\hat{s}_2 = \hat{e}_2$. For any $x \in \{\tilde{x}_\delta^{(i)}\}$ and $y' \in \Omega_1'$ we have $0 \le \theta \le \delta$ and so $0 \le \sin\theta \le \sin\delta$. Therefore, $|\Theta| = |\sin\theta\cos\theta'| \le \delta \le \beta_1/2$ for small enough δ. And, (2.24) holds.

Now, for $0 \le \theta_0 < \pi/2$, replace (2.27) with

$$f(y') = \begin{cases} (-1)^{\mu+\varepsilon_0} f_i \Big[1 - |y' - c_i \hat{e}_\delta|\Big] y'_\delta\,; & y' \in B_1(c_i \hat{e}_\delta),\ y'_\delta \ge 0 \\ 0, & \text{otherwise,} \end{cases} \tag{2.42}$$

where $y'_\delta = y' \cdot ((-1)^M \cos\delta\, \hat{s}_1 - \sin\delta\, \hat{s}_2)$. We align the half balls of the support of f along the unit vector $\hat{e}_\delta$ in the direction $(-1)^M \sin\delta\, \hat{s}_1 + \cos\delta\, \hat{s}_2$ so that $B_1(c_i \hat{e}_\delta) \subset \Omega'_1$.

The rest of the proof for this case follows without serious change, through (2.29).

If $\hat{s}_0 \in \partial\Pi_+$ ($\theta_0 = \pi/2$) then $\hat{s}_1 = \hat{s}_0$ and a subsequence approaches the boundary. As before, for any $\delta > 0$ there is a subsequence of form $\tilde{x}^{(i)}_\delta = \tilde{x}^{(i)} + \delta_i \hat{t}_i$ where $\tilde{x}^{(i)} = a_i \hat{s}_1 + b_i \hat{e}_n,\ 0 < b_i \le a_i, a_i \ge 1, a_i \to \infty$ monotonically, $b_i/a_i \to 0, \{\hat{s}_1, \hat{e}_n, \hat{t}_i\}$ is orthonormal and $0 \le \delta_i \le \delta$. Follow the second part of the sharpness proof, from (2.30).

Let

$$B_\delta = \min\left(\frac{A}{(1+\delta\sqrt{A})^2},\ A - \delta,\ \frac{A}{1+\delta A}\right)$$

then $B_\delta < A$. And, $B_\delta > 1$ if

$$\begin{aligned} 0 \ < \ \delta \ &< \ \min\Big((\sqrt{A}-1)/\sqrt{A},\ A-1,\ (A-1)/A\Big) \\ &= \ (\sqrt{A}-1)/\sqrt{A}. \end{aligned}$$

Without loss of generality, we can take A satisfying the conditions following (2.30) and $0 < \delta < (A-1)/A < 1/2$.

For each $j \ge 1$, let θ' be the angle between y' and $a_j \hat{s}_1$ and θ'_δ the angle between y' and $a_j \hat{s}_1 + \delta_j \hat{t}_j$. We have $\theta' - \delta \le \theta'_\delta \le \theta' + \delta$ so replace (2.31) with

$$\Omega'_2(\hat{y}) = \left\{ y' \in \mathbb{R}^{n-1} \Big| |y'| > 1,\ 0 \le \theta'_\delta < \arccos(1/\sqrt{B_\delta}) - \delta \right\}. \tag{2.43}$$

For each $j \ge 1$, let θ be the angle between $\tilde{x}^{(j)}$ and $\hat{e}_n$ and θ_δ the angle between $\tilde{x}^{(j)}_\delta$ and $\hat{e}_n$. Then

$$\sin\theta_\delta = \frac{|a_j \hat{s}_1 + \delta_j \hat{t}_j|}{|\tilde{x}^{(j)} + \delta_j \hat{t}_j|} \ge \frac{a_j - \delta}{|\tilde{x}^{(j)}| + \delta}.$$

For large enough $|\tilde{x}^{(j)}|$, we have $\sin\theta_\delta \ge a_j/|\tilde{x}^{(j)}| - \delta = \sin\theta - \delta$. It follows from the first component of the definition of B_δ that $\sin\theta \ge 1/\sqrt{B_\delta}$ implies $\sin\theta_\delta \ge 1/\sqrt{A}$.

Write $s = |\tilde{x}^{(j)}|/|y'|$, $s_\delta = |\tilde{x}^{(j)}_\delta|/|y'|$. Then for $y' \in \Omega'_2$, $s - \delta \le s_\delta \le s + \delta$. From the second and third components in the definition of B_δ, $1/B_\delta \le s \le B_\delta$ implies $1/A \le s_\delta \le A$. Hence, we can replace Ω_2 with Ω'_2 and carry out the sharpness proof for $a_j \hat{s}_1 + b_j \hat{e}_n$ with the following changes. In (2.32), replace $a_i \hat{e}_1$ with $a_i \hat{s}_1 + \delta_i \hat{t}_i$. In (2.33) and (2.35), replace y'_1 with $y' \cdot \hat{s}_1$. The rest of the proof, through (2.40), follows with minor changes. ■

The growth estimate on $F_{\lambda,M}[f]$ gives estimates for the solutions of the half space Dirichlet and Neumann problems. The modified kernel introduces a singularity at the origin of the integration space. To avoid integrating f there, a continuous cutoff function that vanishes in a neighbourhood of the origin is used.

Corollary 2.1 *Let* $w\colon \mathbb{R}^{n-1} \to [0,1]$ *be continuous such that* $w(y) \equiv 0$ *when* $|y| \leq 1$ *and* $w(y) \equiv 1$ *when* $|y| \geq 2$. *Let* f *be continuous on* $\mathbb{R}^{n-1}$ *and satisfy* (1.5) *with* $\lambda = n/2 \quad (n \geq 2)$. *The function* $u(x) = D_M[wf](x) + D[(1-w)f](x)$ *satisfies*

$$u \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+) \tag{2.44}$$

$$\Delta u = 0, \quad x \in \Pi_+ \tag{2.45}$$

$$u = f, \quad x \in \partial\Pi_+ \tag{2.46}$$

$$u(x) = o\big(|x|^{M+1} \sec^{n-1}\theta\big); \quad x \in \Pi_+, \quad |x| \to \infty. \tag{2.47}$$

Proof: That u is a classical solution, (2.44), (2.45), (2.46), is contained in Corollary 2 of [18]. To prove (2.47), note that the Theorem gives $D_M[wf](x) = \alpha_n x_n F_{\frac{n}{2},M}[wf](x) = o\big(|x|^{M+1}\sec^{n-1}\theta\big)$. And,

$$\begin{aligned} |D[(1-w)f](x)| &\leq \alpha_n x_n \int\limits_{|y'|<2} |f(y')|\,(|x|-2)^{-n}\,dy' \\ &\leq \alpha_n\, 2^n x_n |x|^{-n} \quad \text{if} \quad |x| \geq 4, \end{aligned}$$

so (2.47) is satisfied. ■

Corollary 2.2 *Let* f *and* w *be as in Corollary 2.1 such that* (1.5) *holds with* $\lambda = (n-2)/2 \quad (n \geq 3)$. *Then* $v(x) = N_M[wf](x) + N[(1-w)f](x)$ *satisfies* (2.45) *and*

$$v \in C^2(\Pi_+) \cap C^1(\overline{\Pi}_+) \tag{2.48}$$

$$\frac{\partial v}{\partial x_n} = -f, \quad x \in \partial\Pi_+ \tag{2.49}$$

$$v(x) = o\left(|x|^M \sec^{n-2}\theta\right); \quad x \in \Pi_+, \quad |x| \to \infty. \tag{2.50}$$

Proof: The growth estimate (2.50) follows from the Theorem:

$$N_M[wf](x) = \frac{\alpha_n}{n-2} F_{\frac{n-2}{2},M}[wf](x).$$

And,

$$\begin{aligned} |N[(1-w)f](x)| &\leq \frac{\alpha_n}{n-2} \int\limits_{|y'|<2} |f(y)|\,(|x|-2)^{2-n}\,dy' \\ &\leq \alpha_n (n-2)^{-1}\, 2^{n-2} |x|^{2-n} \quad \text{if} \quad |x| \geq 4. \end{aligned}$$

Theorem 1 of [10] shows (2.45), (2.48) and (2.49) hold. ■

Remark 2.2 In Corollary 2.1, the solution to (2.44)–(2.47) is unique if $M = 0$ and if $M \geq 1$ it is unique to the addition of a harmonic polynomial of degree M vanishing on $\partial\Pi_+$ ([15], Theorem 3.1). Similarly, in Corollary 2.2, if $M = 0$ the solution to (2.45), (2.48)–(2.50) is unique and if $M \geq 1$ it is unique to the addition of a harmonic polynomial $p(x)$ of degree $M - 1$ that is even about $x_n = 0$. This can be proved by modifying the proof of Theorem 3.1 ([15]): For a function v that satisfies (2.48)–(2.50) with $f = 0$, its even extension is defined in $\mathbb{R}^n$, and by expanding v in terms of spherical harmonics, using (2.50), the conclusion is obtained.

Remark 2.3 If $f(y')\,|y'|^{n-2-M}$ is integrable at the origin then we can use $u(x) = D_M[f](x)$ and $v(x) = N_M[f](x)$ in Corollaries 2.1 and 2.2, respectively.

Corollary 2.3 *If $\omega : \Pi_+ \to (0, \infty)$ then ω is a sharp growth condition for $F_{\lambda,M}$ if and only if there are constants $0 < S < T < \infty$ and $N > 0$ such that $S \leq |x|^{-M} \cos^{2\lambda}\theta\,\omega(x) \leq T$ for all $x \in \Pi_+$ with $|x| > N$.*

Proof: Throughout the proof f will satisfy (1.5) and $|x|$, $|x^{(i)}| > N$.

Suppose S and T exist as above. Then

$$|F_{\lambda,M}[f](x)|\,/\omega(x) \leq |F_{\lambda,M}[f](x)|\,S^{-1}|x|^{-M}\cos^{2\lambda}\theta \to 0$$

so $F_{\lambda,M}[f] = o(\omega)$.

Let $\psi\colon \Pi_+ \to (0,\infty)$ with $\psi = o(\omega)$ then $\psi(x) = o(|x|^M \sec^{2\lambda}\theta)$. Given $\{x^{(i)}\}$ in Π_+ take f as in the proof of the Theorem. Then $F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \not\to 0$. Hence, ω is sharp.

Now suppose ω is sharp. If $\chi(x) := |x|^{-M}\cos^{2\lambda}\theta\,\omega(x)$ is unbounded then there is a sequence $\{x^{(i)}\}$ on which $\chi(x^{(i)}) \to \infty$. Let

$$\psi(x) = \begin{cases} |x|^M \sec^{2\lambda}\theta & \text{on} \quad \{x^{(i)}\} \\ \omega(x)/|x|, & \text{otherwise,} \end{cases}$$

then $\psi = o(\omega)$ but for any f we have $F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \to 0$, since $F_{\lambda,M}[f](x) = o(|x|^M \sec^{2\lambda}\theta)$. This contradicts the assumption that ω was sharp (Definition 2.1, (ii)). Hence T exists as above.

If $\chi \to 0$ on some sequence $\{x^{(i)}\}$ then take f such that

$$\limsup_{i\to\infty} F_{\lambda,M}[f](x^{(i)})|x^{(i)}|^{-M}\cos^{2\lambda}\theta_i \geq 1 \quad (\cos\theta_i = x_n^{(i)}/|x^{(i)}|). \tag{2.51}$$

Then

$$\limsup_{i\to\infty} \frac{F_{\lambda,M}[f](x^{(i)})}{\omega(x^{(i)})} = \limsup_{i\to\infty} \frac{F_{\lambda,M}[f](x^{(i)})}{|x^{(i)}|^M \sec^{2\lambda}\theta_i} \frac{|x^{(i)}|^M \sec^{2\lambda}\theta_i}{\omega(x^{(i)})} = \infty,$$

which contradicts the sharpness assumption (i) of Definition 2.1. Hence, S exists as above. ■

Remark 2.4 The angular blow up predicted for $F_{\lambda,M}$ as $|x| \to \infty$ can be expected to occur only as x approaches $\partial\Pi_+$ within a thin or rarefied set. See [2], [7], [14] and references therein.

3. Representation of the Neumann solution.

The modified kernel $K_M(\lambda, x, y')$ satisfies a differential-difference equation for the derivative with respect to $\theta, |x|, y_i, |y|, x_n$, $|y'|$, y'_i and θ', relating the derivative to $K_M(\lambda+1, x, y')$, $K_{M-1}(\lambda+1, x, y')$ and $K_{M-2}(\lambda+1, x, y')$. The integration of these equations give representations of the modified Neumann integral in terms of the modified Dirichlet integral.

Proposition 3.1 *Let $n \geq 3$, $M \geq 0$, $\lambda > 0$, $x \in \Pi_+$ and $y' \in \mathbb{R}^{n-1}$. Use the convention that $K_m = K$ if $m \leq 0$. Then*

(i) $$\frac{\partial K_M}{\partial \theta}(\lambda, x, y') = 2\lambda\, x_n\, \hat{y}\cdot y'\, K_{M-1}(\lambda+1, x, y')$$

(ii) $$\frac{\partial K_M}{\partial |x|}(\lambda, x, y') = 2\lambda \left[\sin\theta\ \hat{y}\cdot y'\, K_{M-1}(\lambda+1, x, y') - |x| K_{M-2}(\lambda+1, x, y')\right]$$

(iii) $$\frac{\partial K_M}{\partial y_i}(\lambda, x, y') = 2\lambda \left[y'_i\, K_{M-1}(\lambda+1, x, y') - y_i K_{M-2}(\lambda+1, x, y')\right], \ 1 \leq i \leq n-1$$

(iv) $$\frac{\partial K_M}{\partial |y|}(\lambda, x, y') = 2\lambda \left[\hat{y}\cdot y'\, K_{M-1}(\lambda+1, x, y') - |y| K_{M-2}(\lambda+1, x, y')\right]$$

(v) $$\frac{\partial K_M}{\partial x_n}(\lambda, x, y') = -2\lambda\, x_n\, K_{M-2}(\lambda+1, x, y')$$

(vi) $$\frac{\partial K_M}{\partial |y'|}(\lambda, x, y') = 2\lambda \left[y\cdot\hat{y}'\, K_{M-1}(\lambda+1, x, y') - |y'| K_M(\lambda+1, x, y')\right]$$

(vii) $$\frac{\partial K_M}{\partial y'_i}(\lambda, x, y') = 2\lambda \left[y_i\, K_{M-1}(\lambda+1, x, y') - y'_i\, K_M(\lambda+1, x, y')\right], \ 1 \leq i \leq n-1$$

(viii) $$\frac{\partial K_M}{\partial \theta'}(\lambda, x, y') = -2\lambda\, |y|\, |y'| \sin\theta'\, K_{M-1}(\lambda+1, x, y').$$

The proofs rest on the identities

$$\frac{d}{dt} C_m^{\lambda}(t) = 2\lambda\, C_{m-1}^{\lambda+1}(t) \tag{3.1}$$

$$C_0^{\lambda}(t) = 1, \quad C_m^{\lambda}(t) \equiv 0 \quad \text{for} \quad m = 0, -1, -2, \cdots \tag{3.2}$$

$$mC_m^\lambda(t) = 2\lambda[t\, C_{m-1}^{\lambda+1}(t) - C_{m-2}^{\lambda+1}(t)] \tag{3.3}$$

$$(m+2\lambda)C_m^\lambda(t) = 2\lambda[C_m^{\lambda+1}(t) - t\, C_{m-1}^{\lambda+1}(t)] \tag{3.4}$$

([16], 4.7.28). In (iv) x_n is fixed. If θ is held constant for the differentiation then

$$\frac{\partial K_M}{\partial |y|}(\lambda, x, y') = 2\lambda\, [\hat{y}\cdot y'\, K_{M-1}(\lambda+1, x, y') - |x| \csc\theta\, K_{M-2}(\lambda+1, x, y')]\,.$$

This leads to a similar change in (iv) of Proposition 3.2.

Proof of (i): From (1.1), (1.6), (3.1) and (3.2)

$$\begin{aligned}\frac{\partial K_M}{\partial \theta}(\lambda, x, y') &= 2\lambda\, |y'|\, |x| \cos\theta \cos\theta' K(\lambda+1, x, y') \\ &\qquad - 2\lambda \sum_{m=1}^{M-1} |x|^m |y'|^{-(m+2\lambda)} C_{m-1}^{\lambda+1}(\Theta) \cos\theta\cos\theta' \\ &= 2\lambda\, x_n\, |y'| \cos\theta' K_{M-1}(\lambda+1, x, y'). \quad \blacksquare\end{aligned}$$

The other proofs follow in a similar manner from (3.1)–(3.4). Note that $\hat{y}$ and $\Theta = \sin\theta\, \cos\theta' = \sin\theta\; \hat{y}\cdot\hat{y}'$ are independent of $|x|$ and that $\tan\theta = |y|/x_n$ so that $\partial\theta/\partial y_i = y_i\, x_n/(|x|^2|y|)$ and $\partial\theta/\partial x_n = -\sin\theta/|x|$.

Now introduce the following notation. If z_1 and z_2 are in $\mathbb{R}^{n-1}$ then $f_{z_1}(z_2) = z_1 \cdot z_2\, f(z_2)$. If $0 \le s \le \pi/2$ then $x(s)$ indicates x with the polar angle θ replaced by s, i.e., $x(s) = y(s) + x_n(s)\hat{e}_n$, where $y(s) = |x| \sin s\, \hat{y}$, $x_n(s) = |x| \cos s$, $x(\theta) = x$ and $y(\theta) = y$. Note that $|x|$ and $\hat{y}$ are independent of θ. And, if $x = \sum_{j=1}^{n} x_j \hat{e}_j$ then $\breve{x}_i(t) = \sum_{j\neq i} x_j\hat{e}_j + t\hat{e}_i$ $(1 \le i \le n)$.

Integrating (i) through (iv) above and setting $\lambda = (n-2)/2$ we obtain

Proposition 3.2 *Let f be continuous with the origin not in the closure of its support and satisfy* (1.5) *with $\lambda = (n-2)/2$ $(n \ge 3)$. Let $M \ge 0$, $\lambda > 0$, $x \in \Pi_+$ and adopt the convention that $D_m = D$ for $m \le 0$. Then the following are equal to $N_M[f](x)$*

(i) $$\int_{t=\theta_0}^{\theta} D_{M-1}[f_{\hat{y}}](x(t))\, dt + N_M[f](x(\theta_0)), \quad 0 \le \theta_0 \le \tfrac{\pi}{2}$$

(ii) $$\tan\theta \int_{t=r_0}^{|x|} D_{M-1}[f_{\hat{y}}](t\hat{y}) \tfrac{dt}{t} - \sec\theta \int_{t=r_0}^{|x|} D_{M-2}[f](t\hat{x})\, dt + N_M[f](r_0\hat{x}), \quad r_0 \ge 0$$

(iii) $$\frac{1}{x_n}\int_{t=t_i}^{y_i} D_{M-1}[f_{\hat{e}_i}](\breve{x}_i(t))\,dt-\frac{1}{x_n}\int_{t=t_i}^{y_i} D_{M-2}[f](\breve{x}_i(t))\,t\,dt$$

$$+N_M[f](\breve{x}_i(t_i));\quad t_i\in\mathbb{R},\, 1\leq i\leq n-1$$

(iv) $$\frac{1}{x_n}\int_{t=\rho}^{|y|} D_{M-1}[f_{\hat{y}}](t\hat{y}+x_n\hat{e}_n)\,dt-\frac{1}{x_n}\int_{t=\rho}^{|y|} D_{M-2}[f](t\hat{y}+x_n\hat{e}_n)\,t\,dt$$

$$+N_M[f](\rho\hat{y}+x_n\hat{e}_n),\quad \rho\geq 0$$

(v) $$-\int_{t=t_n}^{x_n} D_{M-2}[f](\breve{x}_n(t))\,dt+N_M[f](\breve{x}_n(t_n)),\quad t_n\geq 0.$$

Proof: Integrate each of (i)–(v) in Proposition 3.1 with respect to the relevant variable and set $\lambda=(n-2)/2$. Multiply by $((n-2)/2)f(y')$ and integrate $y'\in\mathbb{R}^{n-1}$. Because of (1.5) the integrals $D_{M-2}[|f|](x)$ and $D_{M-1}[|f_{\hat{y}}|](x)$ converge to continuous functions on $\overline{\Pi}_+$. The same is true for each modified Dirichlet integral in (i)–(v). Fubini's Theorem now justifies the interchange of orders of integration. ■

Remark 3.1 We can relax the condition that f be continuous if we refrain from evaluating $N_M[f]$ on $\partial\Pi_+$. This requires taking $0\leq\theta_0<\pi/2$, $r_0>0$, $\rho>0$ and $t_n>0$. We can dispense with the restriction on the support of f if $f(y')\,|y'|^{-(M+n-3)}$ is integrable at the origin. When $M=0$ there is no restriction on the support of f.

We can use Proposition 3.2(i) to confirm the growth estimate (2.50). If f satisfies (1.5) with $\lambda=(n-2)/2$, $n\geq 3$, and if $0\leq\theta_0<\pi/2$ then (2.19) gives $N_M[f](x(\theta_0))=o(|x|^M)$. From Corollary 2.1, $D_{M-1}[f_{\hat{y}}](x)=o(|x|^M\sec^{n-1}\theta)$. Integrating over θ and using (i) of Proposition 3.2,

$$N_M[f](x)=o(|x|^M\sec^{n-2}\theta)+o(|x|^M)=o(|x|^M\sec^{n-2}\theta),$$

in agreement with Corollary 2.2.

4. Second type of modified kernel.

Using the generating function (2.1) with $z=|y'|/|x|$, $t=\Theta=\sin\theta\cos\theta'$, we can define a second type of modified kernel

$$\tilde{K}_M(\lambda,x,y')=K(\lambda,x,y')-\sum_{m=0}^{M-1}\frac{|y'|^m}{|x|^{m+2\lambda}}C_m^\lambda(\Theta), \tag{4.1}$$

defined for $|x| > 0$ and $M \geq 1$. The convergence condition corresponding to (1.5) is now

$$\int_{\mathbb{R}^{n-1}} |f(y')|(|y'|^{M-1} + 1)\, dy' < \infty. \tag{4.2}$$

If (4.2) is satisfied, define

$$\tilde{F}_{\lambda,M}[f](x) = \int_{\mathbb{R}^{n-1}} f(y')\tilde{K}_M(\lambda, x, y')\, dy'. \tag{4.3}$$

Define $\tilde{D}_M$ and $\tilde{N}_M$ in terms of $\tilde{F}_{\lambda,M}$ as in (1.7) and (1.8). Each $x_n|x|^{-(m+n)}C_m^{n/2}(\Theta)$ in the kernel $\tilde{D}_M$ is harmonic in $\mathbb{R}^n\backslash\{0\}$ (Remark 2.1). The same can be said for $|x|^{-(m+n-2)}C_m^{(n-2/2)}(\Theta)$ in the kernel $\tilde{N}_M$. Hence, $\tilde{D}_M[f]$ and $\tilde{N}_M[f]$ are harmonic in Π_+. Results similar to Propositions 3.1 and 3.2 hold for $\tilde{K}_M$, $\tilde{D}_M$ and $\tilde{N}_M$.

However, $\tilde{D}_M[f]$ is not continuous on $\overline{\Pi}_+$. Since (4.2) implies (1.4) (with $\lambda = n/2$) the unmodified Poisson integral $D[f]$ is continuous for $x_n \geq 0$ if f is continuous. Hence,

$$\tilde{D}_M[f](x) = D[f](x) - \alpha_n x_n \sum_{m=0}^{M-1} |x|^{-(m+n)} \int_{\mathbb{R}^{n-1}} |y'|^m f(y') C_m^{n/2}(\Theta)\, dy'$$

and $\tilde{D}_M[f]$ is continuous for $x_n \geq 0$, $x \neq 0$. Similar remarks apply to the Neumann case. We will work with $\tilde{D}_M$ and $\tilde{N}_M$ only in the limit $|x| \to \infty$.

Growth estimates for $\tilde{F}_{\lambda,M}$ are similar to those for $F_{\lambda,M}$.

Theorem 4.1 *If* (4.2) *holds for measurable* f *then*

$$\tilde{F}_{\lambda,M}[f](x) = o(|x|^{-(M+2\lambda-1)} \sec^{2\lambda}\theta) \quad (x \in \Pi_+,\ |x| \to \infty)$$

and this estimate is sharp in the sense of Definition 2.1.

Proof: Throughout the proof d_1 and d_2 will be positive constants (depending on λ and M). In (2.11) replace $|x|/|y'|$ by $|y'|/|x|$ and in the proof of Theorem 2.1 let $s = |y'|/|x|$.

If $0 < \lambda < \frac{1}{2}$ then (2.15), (2.17) and (2.20) give

$$\begin{aligned} |\tilde{K}_M(\lambda, x, y')| &\leq d_1\, K(\lambda, x, y')\, s^{M-1} \int_{\zeta=0}^{s} |1-\zeta|^{2\lambda-1}\, d\zeta \\ &\leq \frac{d_2\, s^{M+2\lambda-1} \sec^{2\lambda}\theta}{(|x| + |y'|)^{2\lambda}}, \end{aligned}$$

from which $\tilde{F}_{\lambda,M}[f](x) = o(|x|^{-(M+2\lambda-1)} \sec^{2\lambda}\theta)$.

If $\lambda \geq \frac{1}{2}$ then (2.15) and (2.17) give

$$|\tilde{K}_M(\lambda, x, y')| \leq \frac{d_2|y'|^{M-1}\sec^{2\lambda}\theta}{|x|^{M+2\lambda-1}}\left[1-(1+s)^{-2\lambda}\right] \leq \frac{d_2|y'|^{M-1}\sec^{2\lambda}\theta}{|x|^{M+2\lambda-1}}. \tag{4.4}$$

Since $1-(1+s)^{-2\lambda} \to 0$ as $s \to 0$, integrating (4.4) and noting (4.2), dominated convergence gives

$$\int_{\mathbb{R}^{n-1}} f(y')\tilde{K}_M(\lambda, x, y')\,dy' = o(|x|^{-(M+2\lambda-1)}\sec^{2\lambda}\theta) \quad (x \in \Pi_+,\ |x| \to \infty).$$

To prove this sharp, interchange $|x|$ and $|y'|$ in the proof of Theorem 2.1 and proceed in a similar manner. ∎

The modified kernel furnishes an asymptotic expansion of $D[f]$ and $N[f]$.

Theorem 4.2 *Let f be measurable such that* (4.2) *holds for a positive integer M. Then, as $x \to \infty$ in Π_+*

(i) $$D[f](x) = \sum_{m=0}^{M-1} |x|^{-(m+n-1)} Y_{m+1}^{(0)}(\hat{x}) + o(|x|^{-(M+n-2)}\sec^{n-1}\theta) \quad (n \geq 2)$$

(ii) $$N[f](x) = \sum_{m=0}^{M-1} |x|^{-(m+n-2)} Y_{m}^{(1)}(\hat{x}) + o(|x|^{-(M+n-3)}\sec^{n-2}\theta) \quad (n \geq 3)$$

where $Y_m^{(0)}$ is given by (4.5) *below and is a spherical harmonic of degree m that vanishes on $\partial\Pi_+$ and $Y_m^{(1)}$ is given by* (4.6) *below and is a spherical harmonic of degree m whose normal derivative vanishes on $\partial\Pi_+$. The data f can be chosen so that simultaneously the leading order term* ($m = 0$) *does not vanish and the order relation is sharp (in the sense of Definition 2.1).*

Proof: To prove (i) use (4.1) with $\lambda = n/2$, f as in the Theorem and $|x| > 0$,

$$D[f](x) = \alpha_n x_n \int_{\mathbb{R}^{n-1}} f(y') \sum_{m=0}^{M-1} \frac{|y'|^m}{|x|^{m+n}} C_m^{n/2}(\Theta)\,dy' + \tilde{D}_M[f](x).$$

Now (i) follows from Theorem 4.1 and the definition

$$Y_{m+1}^{(0)}(\hat{x}) = \alpha_n \cos\theta \int_{\mathbb{R}^{n-1}} f(y')\,|y'|^m\, C_m^{n/2}(\sin\theta\ \hat{y}\cdot\hat{y}')\,dy'. \tag{4.5}$$

Clearly $Y^{(0)}_{m+1}$ vanishes when $\theta = \pi/2$. It is a spherical harmonic of degree $m+1$ by Remark 2.1.

Given $\psi(x) = o(|x|^{-(M+n-2)}\sec^{n-1}\theta)$ take f as in Theorem 4.1 so that $\tilde{D}_M[f](x) = o(|x|^{-(M+n-2)}\sec^{n-1}\theta)$ is sharp. In particular, f can be taken to be positive for $y_1 > 0$ with a super-even extension (M even) or super-odd extension (M odd) to $y_1 < 0$ ($A_\lambda > 1$ in the proof of Theorem 2.1 (2.34). See Step III in the outline of the proof for an explanation of the terminology "super-even" and "super-odd.") The leading order term in (i) is with $m = 0$, $Y^{(0)}_1(\hat{x}) = \alpha_n \cos\theta \int\limits_{\mathbb{R}^{n-1}} f(y')\,dy'$. With f as above this spherical harmonic does not vanish if $0 \le \theta < \pi/2$. With the definition

$$Y^{(1)}_m(\hat{x}) = \frac{\alpha_n}{n-2} \int\limits_{\mathbb{R}^{n-1}} f(y')\,|y'|^m\, C^{(n-2)/2}_m(\Theta)\,dy', \tag{4.6}$$

the proof of (ii) is similar. ∎

The addition formula for Gegenbauer polynomials can be used to separate the θ dependence in (i) and (ii). First write

$$C^{n/2}_m(\sin\theta\ \hat{y}\cdot\hat{y}') = \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \gamma_{n,m,\ell}(\theta\,)C^{(n-1)/2}_{m-2\ell}(\hat{y}\cdot\hat{y}'),$$

where

$$\gamma_{n,m,\ell}(\theta) = \frac{(n-2)!\,(-1)^\ell\,(2\ell)!\,(n+2m-4\ell-1)\,\Gamma\left(\frac{n}{2}+m-2\ell\right)\,\Gamma\left(\frac{n}{2}+m-\ell\right)}{4^{2\ell-m}\,\Gamma^2(n/2)\,\ell!\,(n+2m-2\ell-1)!} \times$$
$$\times \sin^{m-2\ell}\theta\, C^{n/2+m-2\ell}_{2\ell}(\cos\theta)$$

([6] 10.9.34†, 10.9.19). Then (4.5) becomes

$$Y^{(0)}_{m+1}(\hat{x}) = \alpha_n \cos\theta \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \gamma_{n,m,\ell}(\theta)\,\delta_{n,m,\ell}(\hat{y}),$$

where

$$\delta_{n,m,\ell}(\hat{y}) = \int\limits_{\mathbb{R}^{n-1}} f(y')\,|y'|^m\, C^{(n-1)/2}_{m-2\ell}(\hat{y}\cdot\hat{y}')\,dy'$$

and is independent of $|x|$ and θ.

A similar separation of $|x|$, θ and $\hat{y}$ dependence in (ii) is given by

$$Y^{(1)}_m(\hat{x}) = \frac{\alpha_n}{n-2} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \gamma_{n-2,m,\ell}(\theta)\,\delta_{n-2,m,\ell}(\hat{y}) \quad (n \ge 3).$$

†The first term in the sum over m in this formula should read 2^{2m}.

The function defined by $Z_m(\hat{y}_1, \hat{y}_2) = C_m^{n/2}(\hat{y}_1 \cdot \hat{y}_2)$ is known as a zonal harmonic of degree m with pole $\hat{y}_1 \in \partial B_1$, evaluated at $\hat{y}_2 \in \partial B_1$ (see [5], Chapter 5).

If the integral in (4.2) converges for all $M \geq 1$, letting $M \to \infty$ in Theorem 4.2 will give asymptotic series for $D_M[f]$ and $N_M[f]$. As the following example shows, these series will not in general be convergent.

Example. Let $f(y) = \exp(-|y|)$ and let $d\omega_{n-1}$ be surface measure on the unit ball of $\mathbb{R}^{n-1}$. Then for $n \geq 3$, (4.6) becomes

$$\begin{aligned} Y_m^{(1)}(\hat{x}) &= \frac{\alpha_n}{n-2} \int\limits_{\rho=0}^{\infty} e^{-\rho} \rho^{m+n-2}\, d\rho \int\limits_{\partial B_1} C_m^{(n-2)/2}(\sin\theta\ \hat{y}\cdot\hat{y}')\, d\omega_{n-1} \\ &= \frac{\alpha_n}{n-2}\, (m+n-2)!\, (n-2)\, \omega_{n-2}\, I_{n,m}^{(1)}(\theta), \end{aligned}$$

where

$$I_{n,m}^{(1)}(\theta) = \int\limits_{\phi=0}^{\pi} C_m^{(n-2)/2}(\sin\theta\cos\phi) \sin^{n-3}\phi\, d\phi$$

and the surface integral was evaluated by spherical means [12]. The integral $I_{n,m}^{(1)}(\theta)$ is known ([11] 7.323.2, together with [6] 10.9.19),

$$I_{n,m}^{(1)}(\theta) = \begin{cases} \frac{2^{n-3}\Gamma(n/2-1)\,(-1)^k\,(2k)!\,\Gamma(k+n/2-1)\,C_{2k}^{(n-2)/2}(\cos\theta)}{k!\,\Gamma(2k+n-2)}, & m = 2k \\ 0, \quad m \quad \text{odd.} \end{cases}$$

And,

$$Y_{2k}^{(1)}(\hat{x}) = \frac{2^{n-2}\Gamma(n/2-1)\,(-1)^k\,(2k)!\,\Gamma(k+n/2)\,C_{2k}^{(n-2)/2}(\cos\theta)}{\pi k!}$$

$(Y_{2k+1}^{(1)}(\hat{x}) = 0)$. As $k \to \infty$,

$$C_{2k}^{(n-2)/2}(\cos\theta) \sim \frac{2\,(n/2+2k-2)!\,\sin\left[n\pi/4 - (n/2+2k-1)\,\theta\right]}{(n/2-2)!\,(2k)!\,(2\sin\theta)^{n/2-1}}$$

([16] 8.4.13), so Stirling's approximation shows that for fixed x

$$\sum_{m=0}^{M-1} |x|^{-m} Y_m^{(1)}(\hat{x}) \quad \text{diverges as} \quad M \to \infty.$$

With the Dirichlet expansion we have from (4.5)

$$Y_{m+1}^{(0)}(\hat{x}) = \alpha_n (m+n-2)!\,(n-2)\,\omega_{n-2}\, I_{n,m}^{(0)}(\theta),$$

where

$$I_{n,m}^{(0)}(\theta) = \cos\theta \int_{\phi=0}^{\pi} C_m^{n/2}(\sin\theta\cos\phi)\sin^{n-3}\phi\, d\phi.$$

If $n \geq 5$ then (2.3) and integration by parts give

$$I_{n,m}^{(0)}(\theta) = \frac{1}{(n-2)\sin\theta}\frac{d}{d\theta}I_{n-2,m+2}^{(1)}(\theta)$$

and

$$\sum_{m=0}^{M-1} |x|^{-m} Y_m^{(1)}(\hat{x}) \quad \text{diverges as} \quad M \to \infty. \tag{4.7}$$

When $n = 2$, we use

$$C_m^1(\cos\phi) = \frac{\sin[(m+1)\phi]}{\sin\phi}$$

and replace θ by $\pi/2 - \phi$ (see the end of the proof of Lemma 2.1). With Dirichlet data $f(\xi) = \exp(-|\xi|)$ we have $Y_{m+1}^{(0)}(\hat{x}) = 2\, m! \sin[(m+1)\phi]/\pi$, if m is even and again

$$\sum_{m=0}^{M-1} \frac{m! \sin[(m+1)\theta]}{r^m} \quad \text{diverges as} \quad M \to \infty$$

for fixed $x = r\cos\theta\, \hat{e}_1 + r\sin\theta\, \hat{e}_2$.

When $n = 3$, $I_{3,m}^{(0)}(\theta)$ can be evaluated in terms of Legendre polynomials (since $C_m^{1/2}(t) = P_m(t)$) and when $n = 4$, $I_{4,m}^{(0)}(\theta)$ can be evaluated in terms of trigonometric functions. In both cases, the conclusion of (4.7) remains valid.

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